AN EXACT SOLUTION OF THE ENERGY EQUATION IN A PARTICULAR CASE OF THE MOTION OF A VISCOUS INCOMPRESSIBLE FLUID

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In this paper an exact solution is obtained, in terms of Weierstrassian elliptic functions, of the energy equation for an incompressible viscous fluid with a Prandtl number equal to unity, for the case of the well known [1] plane steady flow between two nonparallel plane walls. Transformation of the solution from Weierstrassian functions to the tabulated Jacobian functions [2] makes it possible to present the solution thus obtained in graphical form.

We note that after this work had been practically completed, the author became aware that this exact solution had already been considered by Millsaps and Pohlhausen [3]. However, in the latter paper the energy equation was integrated numerically.

1. Formulation of the problem. It is well known [1] that the Navier-Stokes equations, written in a polar system of coordinates r, θ, admits of an exact solution in the form

$$v^\circ = \frac{rv_r}{2\nu} = -[3\wp(\theta-\theta_0, g_2, g_3)+1], \qquad v_\theta = 0 \tag{1.1}$$

$$p = -\frac{4\rho\nu^2}{r^2}\left[3\wp(\theta-\theta_0, g_2, g_3)+\frac{1}{2}+\frac{3}{8}g_2\right]+\text{const}$$

where v_r is the radial, and v_θ the tangential component of the vector velocity, p is the pressure, ρ e density, ν the kinematic coefficient of viscosity, $\wp$ the Weierstrassian elliptic function, g_2 and g_3 are the invariants of the function $\wp$, θ_0 is an arbitrary constant which may be a complex number. This solution can be interpreted as the plane flow between two nonparallel walls with an included angle of 2α. Then the arbitrary constants θ_0, g_2 and g_3 are determined from the condition of no slip at the walls and from the condition of the specified discharge Q:

$$\wp(\pm\alpha-\theta_0, g_2, g_3)=-\frac{1}{3}, \qquad \int_{-\alpha}^{\alpha} rv_r(\theta)\,d\theta = Q \tag{1.2}$$

It is not difficult now to verify that the equation of energy, written in the polar system of coordinates, admits of an exact solution in the form

$$T=\frac{144A\sigma\nu^2}{C_v r^2}\,t(\theta) \tag{1.3}$$

where A is the reciprocal of the mechanical equivalent of heat, σ is the Prandtl number, whilst the function $t(\theta)$ has to satisfy the inhomogeneous Lame equation

$$t''(u)=[12\sigma\wp(u)+4\sigma-4]\,t(u)-[\wp^3(u)+\wp^2(u)+l\wp(u)+m] \tag{1.4}$$

where

$$u=\theta-\theta_0, \qquad l=\frac{2}{3}-\frac{1}{4}g_2,\ m=\frac{1}{9}-\frac{1}{4}g_3$$

We shall henceforth consider the case of constant temperature of the walls: then the function $t(u) = t(\theta - \theta_0)$ must satisfy the homogeneous boundary conditions

$$t(\alpha-\theta_0)=t(-\alpha-\theta_0)=0 \tag{1.5}$$

Since the solution of equation (1.4) depends on the function $\gamma(\theta - \theta_0, g_2, g_3)$, defined by the constants θ_0, g_2 and g_3, then in the study of the properties of the solution of equation (1.4) with the boundary conditions (1.5) we shall rely on the properties of the solution (1.1) obtained in the paper [4].

2. The solution for the temperature profile. In the case $\sigma = 1$ the homogeneous Lame equation takes the form:

$$t''(u)=12\wp(u)\,t(u) \tag{2.1}$$

and it follows from the theory of elliptic functions that two of its particular solutions are

$$t_1(u)=\wp'(u) \tag{2.2}$$

$$t_2(u)=\wp'(u)\int\frac{du}{\wp'^2(u)}\equiv-\frac{\wp'(u)}{4}\,[a^2\zeta(u+\omega_1)+b^2\zeta(u+\omega_2)+c^2\zeta(u+\omega_3)+ \\ +(a^2e_1+b^2e_2+c^2e_3)\,u] \tag{2.3}$$

where $\zeta(u)$ denotes the Weierstrassian function, $2\omega_1$ and $2\omega_3$ are the two elementary periods of the function $\gamma(u)$:

$$-\omega_2 = \omega_1 + \omega_3, \qquad e_i = \wp(\omega_i) \qquad (i = 1, 2, 3)$$

$$a = \frac{1}{(e_1 - e_2)(e_1 - e_3)}, \qquad b = \frac{1}{(e_2 - e_1)(e_2 - e_3)}, \qquad c = \frac{1}{(e_3 - e_1)(e_1 - e_2)} \tag{2.4}$$

Applying the method of variation of arbitrary constants and utilizing the well known properties of the Weierstrassian functions $\gamma(u)$ and $\zeta(u)$ for the calculation of the indefinite integrals, we obtain as a particular solution of the inhomogeneous equation (1.4) the following expression:

$$t_0(u) = \wp'(u)\left[\frac{1}{16}\zeta(u) + Bu + C\zeta(u + \omega_1) + D\zeta(u + \omega_2) + G\zeta(u + \omega_3)\right] \tag{2.5}$$

where

$$C = \frac{1}{4}a^2\left(\frac{1}{4}e_1^4 + \frac{1}{3}e_1^3 + \frac{1}{2}le_1^2 + me_1\right), \qquad D = \frac{1}{4}b^2\left(\frac{1}{4}e_2^4 + \frac{1}{3}e_2^3 + \frac{1}{2}le_2^2 + me_2\right.$$

$$G = \frac{1}{4}c^2\left(\frac{1}{4}e_3^4 + \frac{1}{3}e_3^3 + \frac{1}{2}le_3^2 + me_3\right), \qquad B = Ce_1 + De_2 + Ge_3 - \frac{1}{12} \tag{2.6}$$

The general solution of equation (1.4) is then

$$t(\theta) = c_1 t_1(u) + c_2(t_2)(u) + t_0(u) \tag{2.7}$$

where c_1, c_2 are arbitrary constants, which must be found from the conditions (1.5) for the particular solution required with constant temperature of the walls.

We shall henceforth distinguish three separate cases, when the discriminant $\Delta = g_2^3 - 27g_3^2$ of the cubic equation $4s^3 - g_2 s - g_3 = 0$ is less than, greater than, or equal to zero.

1. *The case* $\Delta < 0$. Here we have [4] $\theta_0 = \omega_2$ and the solution for the velocity profile

$$v^\circ = -[3\wp(0 - \omega_2) + 1] = -\left[3H\frac{1 - \mathrm{cn}\,\tau}{1 + \mathrm{cn}\,\tau} + 1 - 2H(2k^2 - 1)\right], \qquad \tau = 2\theta\sqrt{H} \tag{2.8}$$

is the unique solution and gives a purely divergent stream which is symmetric relative to the axis of the diffuser. Here the solution (2.8) depends on two independent parameters H and the Jacobian modular function k^2. The physical parameters of the problem - the Reynolds number $N_{\mathrm{Re}} = Q/2\nu$ and the angle of divergence of the diffuser 2α - are expressed in terms of the parameters H and k^2 by means of the conditions (1.2), and this relation is given by formulas (7.1), (7.3), or the tables presented in paper [4]. If now in the solution (2.7) for the temperature profile we make the substitution $u = \theta - \omega_2$ and transform from the Weierstrassian functions γ and ζ to the Jacobian functions sn, cn, dn and Zn, then the solution for the temperature, satisfying the boundary conditions (1.5), is obtained in the following form:

$$t(\theta) = S(\theta)\ \{\tfrac{1}{16} T(\theta) + B\theta + 2[MP(\theta) - NQ(\theta)] + DR(\theta) -$$
$$- c_1 [2(pP(\theta) - qQ(\theta)) + R(\theta) + \gamma\theta) \qquad (2.9)$$

Here

$$P(\theta) = J(\theta) - \sqrt{H}\, k^2 \frac{\operatorname{sn}\tau \operatorname{cn}\tau}{\operatorname{dn}\tau}, \qquad Q(\theta) = \sqrt{H}\, kk' \frac{\operatorname{sn}\tau}{\operatorname{dn}\tau}$$

$$R(\theta) = J(\theta) + \sqrt{H} \frac{\operatorname{dn}\tau(1 + \operatorname{cn}\tau)}{\operatorname{sn}\tau}, \qquad T(\theta) = J(\theta) - \sqrt{H} \frac{\operatorname{dn}\tau(1 - \operatorname{cn}\tau)}{\operatorname{sn}\tau}$$

$$S(\theta) = 4H^{3/2} \frac{\operatorname{sn}\tau \operatorname{dn}\tau}{(1 + \operatorname{cn}\tau)^2}. \qquad J(\theta) = H\left[2\frac{E}{K} + \frac{2}{3}(2k^2 - 1) - 1\right]\theta + \sqrt{H}\operatorname{Zn}\tau$$

$$\gamma = -(2k^2 - 1) H \left[1 + \frac{8k^2(1 - k^2) + 1}{24k^2(1 - k^2)}\right]$$

$$c_1 = \frac{{}^1/_{16} T(\alpha) + B\alpha + 2[MP(\alpha) - NQ(\alpha)] + DR(\alpha)}{2[pP(\alpha) - qQ(\alpha)] + R(\alpha) + \gamma\alpha}$$

$$M = \tfrac{1}{4}(ps - qt) H^{-4}, \qquad N = \tfrac{1}{4}(pt + qs) H^{-4}$$

$$D = -\tfrac{1}{18}(2k^2 - 1)\ \left[\tfrac{1}{3}(2k^2 - 1)^3 + 4(2k^2 - 1)(1 - k^2)k^2 + \tfrac{4}{9}(2k^2 - 1)^2 H^{-1} -\right.$$
$$\left. - \tfrac{2}{3}(2k^2 - 1) H^{-2} + \tfrac{1}{3} H^{-3}\right]$$

$$B = H\left[(2M - 2D)\tfrac{1}{3}(2k^2 - 1) - 4Nkk' - \frac{1}{12} H^{-1}\right], \qquad k'^2 = 1 - k^2$$

$$p = \frac{8k^2(1 - k^2) - 1}{16k^2(1 - k^2)}, \qquad q = \frac{2k^2 - 1}{4kk'}$$

$$s = H^4\ \{\tfrac{1}{108}(2k^2 - 1)^4 + \tfrac{10}{9}(2k^2 - 1)^2(1 - k^2)k^2 - 4(1 - k^2)^2 k^4 +$$
$$+ \tfrac{1}{9}(2k^2 - 1)\left[\tfrac{1}{9}(2k^2 - 1)^2 - 12(1 - k^2)k^2\right] H^{-1} + \tfrac{1}{3}\left[\tfrac{1}{9}(2k^2 - 1)^2 -\right.$$
$$\left. - 4k^2(1 - k^2)\right] H^{-2} + \tfrac{1}{27}(2k^2 - 1) H^{-3}\}$$

$$t = H^4\{\tfrac{16}{3}(2k^2 - 1)(1 - k^2)k^3k' + \tfrac{2}{3}kk'\left[\tfrac{1}{3}(2k^2 - 1)^2 - 4(1 - k^2)k^2\right]H^{-1} +$$
$$+ \tfrac{4}{9}(2k^2 - 1)kk'H^{-2} + \tfrac{2}{9}kk'H^{-3}\}.$$

K and E are the complete elliptic integrals of the first and second kinds with modulus k.

From the solution (2.9) it is easy to obtain the value of the temperature on the axis of the diffuser. We have

$$t(0) = 2H^2(D - c_1) \qquad (2.10)$$

Since all the functions S, T, P, Q and R occurring in the solution (2.9) are odd, then $t(\theta)$ is an even function, and consequently the distribution of temperature in this case ($\Delta < 0$) is symmetric with respect to the axis.

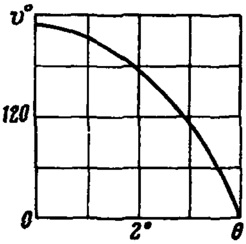

Fig. 1.

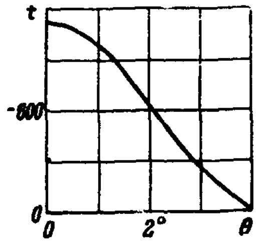

Fig. 2.

In Fig. 1 and 2 are represented the profiles of velocity and temperature, computed from formulas (2.8) and (2.9) for the case $k^2 = 0.9$, $H = 144$, which corresponds to $\alpha = 0.07074 \approx 4^0$, $N_{Re} = 20.34$.

2. *The case* $\Delta > 0$. Here several variants are possible.

First variant: $\theta_0 = \omega_3$. The solution for the velocity profile is [4]

$$v^0 = \frac{rv_r}{2\nu} = -[3\wp(\theta - \omega_3) + 1] =$$

$$= -[3\lambda k^2 \mathrm{sn}^2(\theta\sqrt{\lambda}) - (1 + k^2)\lambda + 1] \tag{2.11}$$

and consequently the stream is symmetric with respect to the axis of the diffuser, the quantity α taking the values η_1, η_2, ..., which are the positive roots of the transcendental equation [4]

$$\mathrm{sn}^2(\eta\sqrt{\lambda}) = \frac{(1 + k^2)\lambda - 1}{3\lambda k^2}$$

In this case the solution (2.11) depends on the two independent parameters λ and the modulus of the Jacobian function k^2. The physical parameters - the number N_{Re} and the angle 2α - are expressed in terms of λ and k^2 by means of the conditions (1.2), and this relation is given by the formulas (8.1), (8.2) or the Tables 2 and 3, presented in the paper [4]. Substituting $\theta - \omega_3$ in the solution (2.7) in place of the argument u, and transforming to Jacobian functions, we obtain the following expression for the temperature profile, satisfying the boundary conditions (1.5), with $\alpha = \eta_1$:

$$t(\theta) = S_2(\theta)\{\tfrac{1}{16}T_2(\theta) + B\theta + CP_2(\theta) + DR_2(\theta) + GQ_2(\theta) - c_2[a^2P_2(\theta) + b^2R_2(\theta) + c^2Q_2(\theta) + \gamma_2\theta]\} \tag{2.12}$$

Here

$$P_2(\theta) = T_2(\theta) - \sqrt{\lambda}k^2\frac{\mathrm{sn}(\theta\sqrt{\lambda})\,\mathrm{cn}(\theta\sqrt{\lambda})}{\mathrm{dn}(\theta\sqrt{\lambda})}, \qquad R_2(\theta) = T_2(\theta) - \sqrt{\lambda}\frac{\mathrm{sn}(\theta\sqrt{\lambda})\,\mathrm{dn}(\theta\sqrt{\lambda})}{\mathrm{cn}(\theta\sqrt{\lambda})}$$

$$Q_2(\theta) = T_2(\theta) + \sqrt{\lambda}\frac{\mathrm{cn}(\theta\sqrt{\lambda})\,\mathrm{dn}(\theta\sqrt{\lambda})}{\mathrm{sn}(\theta\sqrt{\lambda})}, \quad T_2(\theta) = \lambda\left[\frac{E}{K} - \frac{1}{3}(2 - k^2)\right]\theta + \sqrt{\lambda}\,\mathrm{Zn}(\theta\sqrt{\lambda})$$

$$S_2(\theta) = 2\lambda^{3/2}k^2 \operatorname{sn}(0\sqrt{\lambda}) \operatorname{cn}(\theta\sqrt{\lambda}) \operatorname{dn}(\theta\sqrt{\lambda})$$

$$c_2 = \frac{{}^1/_{16}T_2(\alpha) + B\alpha + cP_2(\alpha) + DR_2(\alpha) + EQ_2(\alpha)}{a^2P_2(\alpha) + b^2R_2(\alpha) + c^2Q_2(\alpha) + \gamma_2\alpha} \qquad (2.13)$$

$$C = \frac{2-k^2}{12(1-k^2)^2}\left\{\frac{(2-k^2)^3}{108} - \frac{2-k^2}{108}\varepsilon + \delta + \frac{(2-k^2)^2}{27\lambda} + \frac{2-k^2}{9\lambda^2} + \frac{1}{9\lambda^3}\right\}$$

$$D = \frac{2k^2-1}{12(1-k^2)k^4}\left\{\frac{(2k^2-1)^2}{108} - \frac{2k^2-1}{108}\varepsilon + \delta + \frac{(2k^2-1)^2}{27\lambda} + \frac{2k^2-1}{9\lambda^2} + \frac{1}{9\lambda^3}\right\}$$

$$G = \frac{1+k^4}{12k^4}\left\{\frac{(1+k^2)^3}{108} - \frac{1+k^2}{108}\varepsilon - \delta - \frac{(1+k^2)^2}{27\lambda} + \frac{1+k^2}{9\lambda^2} - \frac{1}{9\lambda^3}\right\}$$

$$B = \frac{\lambda}{3}\left[C(2-k^2) + D(2k^2-1) - G(1+k^2) - \frac{1}{4\lambda}\right]$$

$$\varepsilon = (2-k^2)^2 + (2k^2-1)^2 + (1+k^2)^2, \qquad \delta = \frac{1}{27}(2-k^2)(2k^2-1)(1+k^2)$$

$$a^2 = \frac{1}{(1-k^2)^2\lambda^4}, \qquad b^2 = \frac{1}{(1-k^2)^2k^4\lambda^4}, \qquad c^2 = \frac{1}{k^4\lambda^4}$$

$$\gamma_2 = \frac{1}{3\lambda^3}\left[\frac{2-k^2}{(1-k^2)^2} + \frac{2k^2-1}{(1-k^2)^2} - \frac{1+k^2}{k^4}\right]$$

The temperature on the axis of the stream is

$$t(0) = 2\lambda^2k^2(G - c_2c^2) \qquad (2.14)$$

By virtue of the fact that all the functions occurring in (2.12) are odd, the temperature distribution in this case ($\Delta > 0$, $\theta_0 = \omega_3$ is symmetric with respect to the axis of the diffuser.

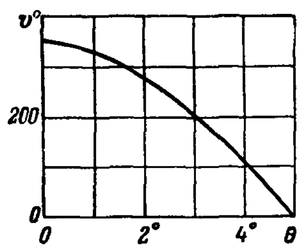

Fig. 3.

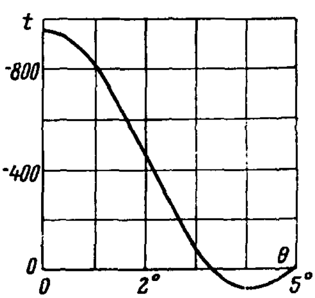

Fig. 4.

In Fig. 3 and 4 are shown the velocity and temperature profiles, computed from formulas (2.11) and (2.12) for the case $k^2 = 0.8$, $\lambda = 196$, which corresponds to $\alpha = 0.08727 \approx 5^\circ$, $N_{Re} = 34.82$. In this case the flow obtained is purely divergent.

With $\alpha = \eta_2 = 2\omega_1 - \eta_1$ the central diverging stream is flanked symmetrically on both sides by regions of inflow (Fig. 5). The temperature in this case is computed again from formula (2.12), but with another value of the constant c_2, which is easily obtained from the following property of the functions T_2, P_2, R_2 and Q_2, e.g.:

$$T_2(\theta \pm 2\omega_1 n) = T_2(\theta) \pm 2n\omega_1\lambda\left[E/K - \tfrac{1}{3}(2-k^2)\right]. \tag{2.15}$$

where n is an integer. Then

$$T_2(\alpha) = T_2(\eta_2) = 2\omega_1\lambda\left[E/K - \frac{1}{3}(2-k^2)\right] - T_2(\eta_1)$$

$T_2(\eta_1)$ has already been computed for the case when $\alpha = \eta_1$.

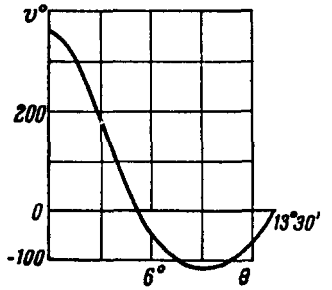

Fig. 5.

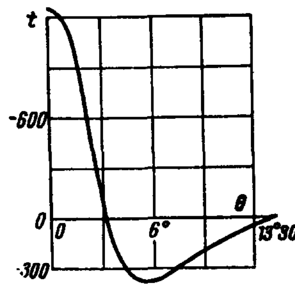

Fig. 6.

In Fig. 5 and 6 are represented the velocity and temperature profiles, computed in this case ($\alpha = \eta_2 = 2\omega_1 - \eta_1$) for $k^2 = 0.8$, $\lambda = 196$, ($\alpha = 0.2346 \approx 13^\circ 30'$, $N_{Re} = 10.82$).

When $\alpha = \eta_3 = 2\omega_1 + \eta_1$ the central diverging stream is flanked symmetrically on both sides by regions of inflow and outflow. The temperature profile is also correspondingly complex.

Second variant: $\theta_0 = \omega_2$. The solution for the velocity and temperature is

$$v^\circ = -[3\wp(\theta - \omega_2) + 1] = -\left[3\lambda k^2 \frac{\mathrm{cn}^2(\theta\sqrt{\lambda})}{\mathrm{dn}^2(\theta\sqrt{\lambda})} - (1+k^2)\lambda + 1\right] \tag{2.16}$$

$$t(\theta) = S_3(\theta)\{\tfrac{1}{16}T_3(\theta) + B\theta + CP_3(\theta) + CR_3(\theta) + GQ_3(\theta) -$$

$$- c_3[a^2P_3(\theta) + b^2R_2(\theta) + c^2Q_3(\theta) + \gamma_2\theta]\}, \quad t(0) = -2\lambda^2k^2(1-k^2)(D - c_3b^2) \tag{2.17}$$

where

$$P_3(\theta) = T_2(\theta), \qquad R_3(\theta) = Q_2(\theta), \qquad Q_3(\theta) = R_3(\theta), \qquad T_3(\theta) = P_2(\theta)$$

$$S_3(\theta) = -2\lambda^{3/2} k^2 (1-k^2) \frac{\operatorname{sn}(\theta\sqrt{\lambda})\operatorname{cn}(\theta\sqrt{\lambda})}{\operatorname{dn}^3(\theta\sqrt{\lambda})}$$

The expression for the constant c_3 is obtained from formula (2.13), if instead of the functions with index 2 we substitute the functions with index 3. The expression for the coefficients a^2, b^2, c^2 and C, D, G, B remain the same as in the first variant.

Here α can assume the following system of values [4]: $\alpha = \omega_1 - \eta_1$, $\alpha = \omega_1 + \eta_1$, $\alpha = \omega_1 + \eta_2$, $\alpha = \omega_1 + \eta_3$ and so on, $\alpha = n\,\omega_1$ (n is a positive integer). If $\alpha = \omega_1 - \eta_1$, then the stream is purely converging and the temperature inside the stream is higher than the temperature of the walls (Fig. 7 and 8), $k^2 = 0.9$, $\lambda = 400$, $\alpha = 4^0$, $N_{\mathrm{Re}} = 31.9$. If $\alpha = \omega_1 + \eta_1$, then the central converging stream is flanked symmetrically by regions of outflow and the temperature in the region of the converging stream is higher than the temperature of the walls, whilst in the region of the diverging stream it is lower than the temperature of the walls (Fig. 9 and 10), $k^2 = 0.8$, $\lambda = 400$, $\alpha = 10^0$, $N_{\mathrm{Re}} = 33$. When $\alpha = \omega_1 + \eta_2$, the central converging stream is flanked symmetrically on both sides by regions of outflow and inflow. When $\alpha = n\,\omega_1$, the solution for the velocity profile has the form [4]:

$$v_r = -\frac{2\nu}{r}\left[3\wp(\theta - \alpha_0 - \omega_3) + 1\right]$$

where $\alpha_0 = \pm(\omega_1 - \eta_1)$ for $\alpha_1 = \omega_1$, and $\alpha_0 = \pm\eta_1$ for $\alpha_2 = 2\omega_1$, and so on. In this case the stream is not symmetric relative to the axis, since

$$\wp(-\theta - \alpha_0 - \omega_3) = \wp(\theta + \alpha_0 + \omega_3) = \wp(\theta + \alpha_0 - \omega_3) \neq \wp(\theta - \alpha_0 - \omega_3)$$

It is easy to show that in this case the temperature profile is also asymmetric.

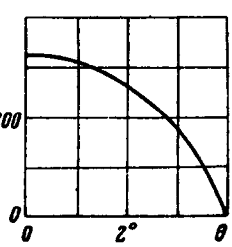

Fig. 7.

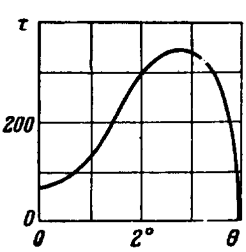

Fig. 8.

3. *The case* $\Delta = 0$. Study of this limiting case shows that here only a single type of profile is possible - this is parabolic. The temperature distribution is described by a parabola of the fourth degree.

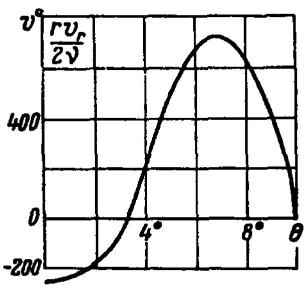

Fig. 9.

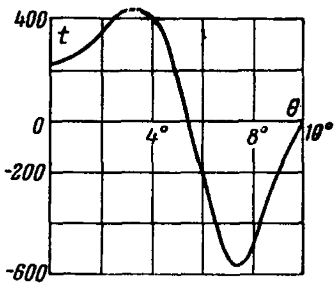

Fig. 10.

From the exact solution obtained above for the temperature profile it follows that a symmetric distribution of the velocity profile corresponds to a symmetric distribution of temperature. An asymmetric velocity profile corresponds to an asymmetric temperature profile. For greater values of N_{Re} there occur local elevations of temperature, in places with extreme velocity gradients, on account of dissipation. In all cases of flow the temperature profile is directed opposite to the velocity profile.

Note. For Prandtl number $\sigma \neq 1$ the equation (1.4) cannot be integrated in closed form. However, if $12\sigma = n\ (n + 1)$ (n is an integer), the general solution of the homogeneous equation (1.4) is obtained in terms of the Weierstrassian functions $\zeta(u)$ and $\sigma(u)$. A particular solution of the inhomogeneous equation (1.4) is now expressed by quadratures in terms of unevaluated integrals, and therefore the general solution of the inhomogeneous equation (1.4) is obtained in a form which is cumbersome for study.

If n = 6, 8, 10, ..., which corresponds to σ = 3.5, 6, 9.16, a particular solution of equation (1.4) can be found in the form

$$t_0 = A_1 \wp^{\frac{n}{2}}(u) + A_2 \wp^{\frac{n}{2}-1}(u) + \dots$$

and the general solution of equation (1.4) is then obtained in closed form. However, in view of its complexity we shall not embark upon a study of it.

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